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Stationary vortices in three-dimensional quasi-geostrophic shear flow

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An existence theorem for localized stationary vortex solutions in an external shear flow is proved. The flow is three-dimensional and quasi-geostrophic in an unbounded domain. The external flow is unidirectional, with linear horizontal and vertical shear. The flow conserves an infinite family of Casimir integrals. Flows that have the same value of all Casimir integrals are called *isovortical flows*, and the potential vorticity (PV) fields of isovortical flows are *stratified rearrangements* of one another. The theorem guarantees the existence of a maximum-energy flow in any family of isovortical flows that satisfies the following conditions: the PV-anomaly must have compact support, it must have the same sign everywhere, and this sign must be the same as the sign of the external horizontal shear over the vertical interval to which the support of the PV-anomaly is confined. This flow represents a stationary and localized vortex, and the maximum-energy property implies that the vortex is stable. The PV-anomaly decreases monotonically outward from the vortex centre in each horizontal plane, but apart from this the profile is arbitrary.

1. Introduction

Coherent vortices are common in most large-scale geophysical flows, particularly in regions of strong shear. In such regions, the vorticity anomaly of the vortices almost invariably has the same sign as the shear of the background flow ('cooperative shear'). Many examples of this are given by the long-lived vortices found in the zonal flow on the giant planets.

It has also been demonstrated in many laboratory experiments and numerical simulations that such vortices can be generated by shear flow instabilities, and that they have a long life-time (sometimes infinite), maintaining themselves by merger with smaller vortices of the same sign. Vortices in 'adverse shear' (i.e. with opposite signs of the background shear and the vorticity anomaly), on the other hand, are rarely seen in real flows or numerical simulations. Yet there exist theoretical solutions describing stationary and linearly stable vortices in adverse shear (Moore & Saffman 1971). In these explicit solutions, however, the background shear is much smaller than the vorticity anomaly.

One explanation of the difference between cooperative and adverse shear is provided by the existence theorem of Nycander (1995) for two-dimensional flow. This theorem states that in every family of 'isovortical flows' (to be defined below) that consists of a background linear shear flow and a compact region of additional vorticity with the same sign as the background shear, there exists a maximum-energy flow, representing a localized and stationary vortex. The vorticity decreases monotonically outward from the vortex centre (assuming that the shear and the vorticity anomaly are positive). Such a vortex is a maximum-energy state, which guarantees that it is stable both linearly and (albeit in an informal sense) nonlinearly. Nothing could be proved about the existence of a stationary vortex in adverse shear, but it is clear from the proof that if such a solution exists, it corresponds to a saddle point of the energy. It can therefore be expected to be unstable, at least nonlinearly.

Another explanation is that a stationary vortex in cooperative shear is a 'maximum entropy state', according to the statistical-mechanical theory of Miller (1990) and Robert & Sommeria (1991). However, the underlying mathematical structure explaining this is again the fact that it is also a maximum-energy state.

These theories apply to ideal two-dimensional flow governed by the Euler equation, which is a highly simplified model of geophysical flows. In the present paper we extend the existence theorem of Nycander (1995) to three-dimensional quasi-geostrophic flow, which is a more realistic model. In this model the stream function for the horizontal velocity field is obtained from the potential vorticity (PV) field by inversion of a three-dimensional linear elliptic operator. The PV is a Lagrangian invariant (i.e. it is conserved along fluid trajectories), which implies the conservation of an infinite family of Casimir integrals (whose integrands are functions of z and the PV). Flows that have the same value of all Casimirs are called *isovortical flows*. We also call the PV-fields of such isovortical flows *stratified rearrangements* of each other. A stratified rearrangement may be generated by a horizontal incompressible deformation of the PV-field that preserves the area inside any contour line of PV at any fixed height level. This is illustrated in figure 1.

We assume the background flow to be unidirectional, with linear horizontal and vertical shear. We then superimpose on this flow a compact region of additional PV ('PV-anomaly'), with the same sign as the background horizontal shear. We will prove that in the set of stratified rearrangements of such a given flow, there exists a maximum-energy flow. This energy maximizer is a localized stationary vortex. As in the case of two-dimensional flows, the fact that this flow maximizes the energy also implies that it is stable (in the context of the quasi-geostrophic model).

Usually, the three-dimensional quasi-geostrophic equation is studied in a domain which is bounded vertically. However, we have not been able to prove the existence theorem for this case, and instead assume that there are no boundaries. Effectively, this means that we study vortices that are small compared to the height of the atmosphere or the ocean. The difficulty with the bounded case appears to be technical, and we believe that the corresponding theorem is valid for that case as well.

The article is organized as follows. In § 2 the basic equations and invariants are given, and a simple heuristic argument for the existence theorem is presented. In § 3 we present the notation and the central theorem to be proved (Theorem 1), and also give an outline of the proof. Section 4 contains some basic theory and inequalities concerning rearrangements, and some theory of convex sets. In § 5 we prove some inequalities that are needed later to prove that the energy maximizer has finite extent. Section 6 contains the proof of Theorem 1. In § 7 we discuss possible generalizations of the theory and its relation to recent numerical simulations of three-dimensional quasi-geostrophic turbulence. The Appendix contains extensions of some standard results on spaces of rearrangements to the stratified case; these are needed for the proof of Theorem 1.

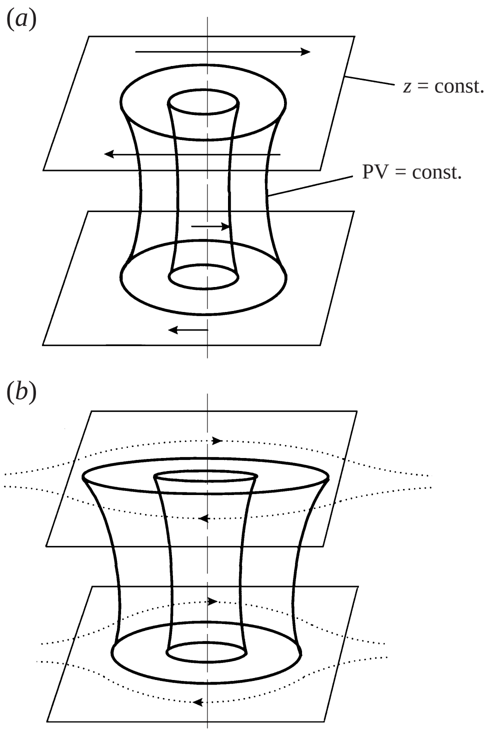

FIGURE 1. Illustration of a stratified rearrangement. In (*a*) a cylindrical region of positive PV-anomaly is shown. The heavy curves depict two surfaces of constant PV, and the intersection between these surfaces and two planes z = const. The arrows show the background flow: a unidirectional flow with linear shear. In (*b*) the PV field has been deformed in such a way that the area inside any curve PV = const. on any plane z = const. is kept constant. Hence it is a stratified rearrangement of the field in (*a*). Theorem 1 proves that it is possible to find a steady vortex solution in the given background flow by such a deformation. In the stationary solution in (*b*), but not in (*a*), the curves PV = const, z = const. are also streamlines. Note that the vortex is more elongated where the background shear is stronger. The dotted curves in (*b*) denote the separatrix.

2. Basic equations and heuristic argument

Three-dimensional quasi-geostrophic flow is described by the equation (Pedlosky 1987)

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\Delta_{\perp} p + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial p}{\partial z}\right)\right] = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_g \cdot \nabla, \end{aligned}\right\}, \qquad (1)$$

where $\Delta_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2$, p is the pressure, $\boldsymbol{v}_g = \rho^{-1}\hat{\boldsymbol{z}} \times \nabla p$ is the geostrophic velocity, and the quantity in square brackets is the potential vorticity (PV), which is a Lagrangian invariant of the flow, advected by the velocity $\boldsymbol{v}_g$. Physically, the PV is conserved on isentropic surfaces rather than on height surfaces, but in the quasi-geostrophic limit they coincide, since the vertical motion is neglected.

We assume that the domain of the flow is infinite in all directions. The existence of solutions of (1) for the vertically bounded case, with suitable boundary conditions at

$z = 0$ and $z = H$, was proved by Bourgeois & Beale (1994). They also showed that solutions of the primitive equations converge to quasi-geostrophic solutions in the limit of small Rossby number, provided that the primitive equations are initialized so that fast solutions are suppressed.

We will neglect the latitudinal dependence of the Coriolis parameter f. For simplicity, we will also assume the buoyancy frequency N to be constant, which does not principally alter the character of the problem. With these assumptions, equation (1) can be written

$$\frac{\partial}{\partial t}\Delta\Psi + J(\Delta\Psi, \Psi) = 0, \tag{2}$$

where Δ is the three-dimensional Laplacian, the Jacobian is defined by $J(f,g) = \partial_x f \partial_y g - \partial_y f \partial_x g$, Ψ is the stream function, the flow being given by $\boldsymbol{v} = \nabla\Psi \times \hat{\boldsymbol{z}}$, and $-\Delta\Psi$ is the PV. The dimensionless variables have been chosen so that the ratio between the vertical and horizontal length scales is f/N.

We now assume that the background flow is given by $\boldsymbol{V} = -2y(c_0 + c_1 z)\hat{\boldsymbol{x}}$, corresponding to the stream function $-(c_0 + c_1 z)y^2$ and the PV $2(c_0 + c_1 z)$. Here c_0 and c_1 are arbitrary constants. This represents a unidirectional flow, with linear horizontal and vertical shear. Decomposing the total stream function as $\Psi = -(c_0 + c_1 z)y^2 + \psi$, equation (2) can be written

$$\frac{\partial}{\partial t}\Delta\psi + J\left(\Delta\psi, -(c_0 + c_1 z)y^2 + \psi\right) = 0, \tag{3}$$

which is the equation we will study in what follows. The PV-anomaly $q = -\Delta\psi$ is assumed to have compact support.

Equation (3) conserves the total energy,

$$E(q) = W(q) - J(q), \tag{4}$$

where

$$W(q) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{q(\boldsymbol{r})q(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}',$$

$$J(q) = \int_{\mathbb{R}^3} (c_0 + c_1 z)y^2 q(\boldsymbol{r})\mathrm{d}\boldsymbol{r}.$$

We call W the *perturbation energy*, since it is quadratic in the vortex PV-anomaly q. Note that W is not conserved by the flow.

Equation (3) further conserves the infinite family of Casimir integrals,

$$C_F = \int_{\mathbb{R}^3} F(z, q)\mathrm{d}\boldsymbol{r},$$

where F is an arbitrary function of both arguments. We call PV-fields $q(\boldsymbol{r})$ that have the same value of all Casimirs *stratified rearrangements* of each other. (A mathematical definition is given in § 3.1.) We may think of the mapping between two stratified rearrangements as a horizontal incompressible deformation that preserves the area inside any contour line of q at any fixed value of z (although slightly more general rearrangements are possible). The corresponding flows are called *isovortical.*

Stationary solutions of equation (3) are given by $J(\Delta\psi, \Psi) = 0$, which expresses a functional dependence between $\Delta\psi$ and Ψ. They can also be obtained formally from the following variational property. A general isovortical first-order perturbation of a given PV-field q (i.e. one satisfying $\delta C_F = 0$ for any F) is given by $\delta q = J(\xi, q)$, where

$\xi(\boldsymbol{r})$ is arbitrary. The variation of the energy caused by such a perturbation is $\delta E = -\int \Psi J(\xi, q)\mathrm{d}\boldsymbol{r} = -\int \xi J(q, \Psi)\mathrm{d}\boldsymbol{r}$. Hence, if $\delta E = 0$ for any ξ, then $J(\Delta\psi, \Psi) \equiv 0$. In particular, a flow that maximises the energy in the set of all stratified rearrangements of some given PV-field q must be stationary. The purpose of the present work is to prove that such an energy maximizer exists, and to give an exact derivation of the steady-state equation. For the proof to be valid it is necessary that q has the same sign everywhere, and that this sign is the same as the sign of the external vorticity $2(c_0 + c_1 z)$ at all height levels where $q \neq 0$.

We first give a simple intuitive argument. If we change the sign of the expression (4), it has exactly the same form as the gravitational potential energy of some mass distribution with the density q. The first term W then represents the interaction energy between the mass elements, and the second term J the contribution from an external gravitational field. Arbitrary stratified rearrangements are obtained by displacing the mass elements horizontally, assuming that the matter in incompressible. No vertical displacement is allowed.

If $c_0 = c_1 = 0$ (i.e. in the absence of external flow) the minimum potential energy is obviously attained by putting the densest matter at the centre at each height level $z =$ const. The corresponding flow is an axisymmetric vortex $q(r, z)$, with q being a monotonic decreasing function of $r = (x^2 + y^2)^{1/2}$, and $q \geqslant 0$ everywhere (or monotonic increasing and $q \leqslant 0$ everywhere). The functional dependence on z is determined by the given vertical distribution, and in principle arbitrary. Such a vortex is trivially stationary, and the present consideration demonstrates that it is also a maximum-energy flow. This helps explain the tendency toward horizontal axisymmetrization and vertical alignment of the vortices that has been seen in recent numerical simulations of three-dimensional quasi-geostrophic turbulence (McWilliams 1989; Viera 1995; Sutyrin, McWilliams & Saravanan 1998). As discussed in § 7, it is typical for many nonlinear infinite-dimensional systems that conditional extrema of conserved quantities act as attractors.

For non-zero external flow, the term $J(q)$ in equation (4) means that the matter is placed in a one-dimensional external potential well, with the minimum at $y = 0$ if $c_0 + c_1 z$ is positive. One intuitively expects that a state of minimum potential energy then still exists, with the densest matter near $y = 0$. This would correspond to a vortex with monotonic radial profile of potential vorticity, in this case flattened in the y-direction, i.e. elongated in the direction of the external flow. Below, we will present a rigorous proof for this conjecture.

3. Statement of results

3.1. *Notation and terminology*

Throughout, *measure* will refer to Lebesgue measure on $\mathbb{R}^N$, and will be called *area* in dimension 2, or *volume* in dimension 3. If $S \subset \mathbb{R}^N$ is measurable then $|S|$ will denote the measure of S.

When f and g are real integrable functions defined on a bounded measurable set $\Omega \subset \mathbb{R}^N$, we say f is a *rearrangement* of g if

$$|\{\boldsymbol{r} \in \Omega : f(\boldsymbol{r}) \geqslant s\}| = |\{\boldsymbol{r} \in \Omega : g(\boldsymbol{r}) \geqslant s\}| \quad \text{for all } s \in \mathbb{R}.$$

A definition of rearrangements on unbounded domains makes most sense for one-signed functions. We say $f : \mathbb{R}^N \to \mathbb{R}$ is *admissible* if f is measurable, non-negative almost everywhere, and satisfies $|\{\boldsymbol{r}\in\mathbb{R}^N\colon f(\boldsymbol{r})>s\}|<\infty$ for all $s>0$. Two admissible functions f and g defined on $\mathbb{R}^N$ will be called *rearrangements* of each other if

$$|\{\boldsymbol{r}\in\mathbb{R}^N\colon f(\boldsymbol{r})\geqslant s\}| = |\{\boldsymbol{r}\in\mathbb{R}^N\colon g(\boldsymbol{r})\geqslant s\}| \quad \text{for all } s>0.$$

When f is square-integrable on bounded measurable $\Omega\subset\mathbb{R}^N$, the set of all rearrangements of f on Ω is denoted $\mathscr{R}_\Omega(f)$, and the closed convex hull in $L^2(\Omega)$ of $\mathscr{R}_\Omega(f)$ is denoted $\mathscr{C}_\Omega(f)$ (and later plays an important technical rôle; see § 4.6 for the definition). We will frequently omit subscript Ω when the discussion of these and other concepts is mathematically informal.

Consider a bounded measurable $\Omega\subset\mathbb{R}^3$ and $q_0\in L^2(\Omega)$. Now $q_0(\cdot,z)$ is square-integrable on $\Omega(z):=\{(x,y)\in\mathbb{R}^2\colon (x,y,z)\in\Omega\}$ for almost every real z. Hence we can define

$$\mathfrak{R}_\Omega(q_0)=\{q\in L^2(\Omega)\colon q(\cdot,z)\in\mathscr{R}_{\Omega(z)}(q_0(\cdot,z)) \quad \text{for a.e. real } z\},$$

$$\mathfrak{C}_\Omega(q_0)=\{q\in L^2(\Omega)\colon q(\cdot,z)\in\mathscr{C}_{\Omega(z)}(q_0(\cdot,z)) \quad \text{for a.e. real } z\}$$

and we refer to elements of $\mathfrak{R}_\Omega(q_0)$ as *stratified rearrangements* of q_0.

To extend the definition to functions on the unbounded domain $\mathbb{R}^3$, for non-negative functions q, $q_0\in L^2(\mathbb{R}^3)$ having compact support, we say q is a *stratified rearrangement* of q_0 if $q(\cdot,z)$ is a rearrangement of $q_0(\cdot,z)$ for almost every real z.

We write points in $\mathbb{R}^3$ as $\boldsymbol{r}=(x,y,z)$, $\boldsymbol{r}'=(x',y',z')$ and so on, abbreviating the volume element to $\mathrm{d}\boldsymbol{r}=\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ where convenient. We fix positive constants c_0 and c_1. For non-negative $q\in L^2(\mathbb{R}^3)$ having compact support, we define

$$Kq(\boldsymbol{r})=\frac{1}{4\pi}\int_{\mathbb{R}^3}|\boldsymbol{r}-\boldsymbol{r}'|^{-1}q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'=\psi(\boldsymbol{r}) \quad \text{for all } \boldsymbol{r}\in\mathbb{R}^3,$$

$$W(q)=\frac{1}{2}\int_{\mathbb{R}^3}q(\boldsymbol{r})Kq(\boldsymbol{r})\mathrm{d}\boldsymbol{r}=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi|^2,$$

where the second form for $W(q)$ follows from the Divergence Theorem, since $\psi(\boldsymbol{r})=O(|\boldsymbol{r}|^{-1})$ and $\nabla\psi(\boldsymbol{r})=O(|\boldsymbol{r}|^{-2})$ as $|\boldsymbol{r}|\to\infty$.

The energy $E=W-J$ is defined in equation (4). Our main result is the following:

THEOREM 1. *Let* $0<z_0<z_1$ *and let* $q_0\in L^2(\mathbb{R}^3)$ *be non-negative and have compact support lying in* $z_0<z<z_1$. *Let* c_0 *and* c_1 *be positive numbers. Then there exists a maximizer* $\bar{q}$ *for E relative to the stratified rearrangements of* q_0 *on* $\mathbb{R}^3$, *and* $\psi:=K\bar{q}$ *satisfies*

$$-\Delta\psi(x,y,z)=\varphi\left(\psi(x,y,z)-(c_0+c_1z)y^2,z\right) \ \text{a.e. in } \mathbb{R}^3$$

for some function $\varphi\colon\mathbb{R}^2\to[0,\infty)$ *such that* $\varphi(\cdot,z)$ *is increasing for almost every real z.*

Remark. Moreover $\bar{q}$ can be assumed doubly Steiner-symmetric; for the definition see § 4.3.

3.2. *Outline of proof of Theorem* 1

A complete proof of Theorem 1 will be given in § 6, but since a number of preliminaries are required, we digress at this stage to explain the strategy, which is modelled on the plan sketched by Benjamin (1976) in his theory of steady vortex-rings.

The first step is to prove the existence of a maximizer for E relative to the stratified rearrangements of q_0 defined on a bounded box Ω. Here the arguments of Benjamin prove difficult to realize in detail, and we follow instead the approach of Burton (1987*a*, Theorem 7). A weak compactness argument is employed, but since the set $\mathfrak{R}_\Omega(q_0)$ is not weakly compact in general, we extend the class of admissible functions for our maximization. We work in the set $\mathfrak{C}_\Omega(q_0)$, which is closed, bounded and convex in $L^2(\Omega)$ and therefore weakly compact, in the sense that any sequence in $\mathfrak{C}_\Omega(q_0)$ has a sub-sequence converging weakly in $L^2(\Omega)$ to an element of $\mathfrak{C}_\Omega(q_0)$. This weak compactness, together with the weak continuity of the energy E, easily leads to the existence of an energy maximizer $\bar{q}$ in the class $\mathfrak{C}_\Omega(q_0)$. To complete the first step, we have to show that $\bar{q}$ in fact belongs to $\mathfrak{R}_\Omega(q_0)$. To this end, the necessary condition at the maximizer $\bar{q}$ is studied, and is found to require that $\bar{q}$ be the unique maximizer of a certain linear functional (defined in terms of $\bar{q}$) relative to $\mathfrak{C}_\Omega(q_0)$. Lemma 6 (see the Appendix) shows that the supremum of any bounded linear functional relative to $\mathfrak{C}_\Omega(q_0)$ is attained by at least one element of $\mathfrak{R}_\Omega(q_0)$. Hence $\bar{q} \in \mathfrak{R}_\Omega(q_0)$. From a geometric viewpoint, one should think of elements of $\mathfrak{R}_\Omega(q_0)$ as 'extreme points' or 'vertices' of $\mathfrak{C}_\Omega(q_0)$.

The mathematics of this first step is more abstract than in the corresponding proof for two-dimensional flow by Nycander (1995). In that case the fact that the maximizer must be symmetric decreasing in x and y could be used to prove that a maximising sequence of rearrangements is totally bounded, and that the sequence is therefore strongly convergent. In the present three-dimensional case, however, the rearrangements in a maximising sequence may oscillate rapidly in z (this is possible even if they are symmetric decreasing in x and y), and the sequence is therefore not totally bounded *a priori*. The weak compactness argument is therefore necessary.

The second step is to show that increasing the size of the confining box Ω indefinitely does not affect the maximizer, i.e. that the support of the maximizer does not touch the boundary of the confining box if the latter is large enough.

Since the contribution $K\bar{q}$ to the stream function from the vortex vanishes at infinity, the streamline $\Psi = 0$ for fixed z comes arbitrarily close to the y-axis for $|x| \to \infty$. This streamline is therefore a separatrix. Inside it the streamlines are closed, and outside they are open. (This is an important difference between the present case and the two-dimensional problem treated by Nycander 1995.) In that case the vortex contribution to the stream function diverges logarithmically at infinity. There is therefore no separatrix, and all streamlines are closed. The same is true for three-dimensional quasi-geostrophic flow in a domain which is bounded vertically.)

From the far-field behaviour of Ψ it is possible to show that the area inside the separatrix at any fixed z is unbounded (i.e. that it can be made arbitrarily large by increasing the size of the box, cf. Lemma 4 in § 5). To estimate the far-field behaviour we first show that the maximizer must have positive energy, cf. Lemma 1 in § 5, and that as a consequence of this the volume of its support must be finite in some finite box, cf. Lemma 3 in § 5. Together with the necessary condition for a maximum, which says that $\bar{q}$ is an increasing function of the stream function $\Psi := K\bar{q} - (c_0 + c_1 z)y^2$ for (almost every) fixed z, the unbounded area inside the separatrix implies that the support of the maximizer lies entirely in the interior of the box, if the latter is large enough. (Lemma 2 in § 5 is used to give an upper estimate of the necessary size of the box.) Hence, if we choose Ω large enough for fixed q_0, the maximizer $\bar{q}$ is also a maximizer for all larger domains Ω, thus completing the proof.

Noteworthy features of the method are that no smoothness of q_0 is assumed (hence vortex patches can be treated), and that the variations performed in deriving the steady-state equation are exact rather than first-order approximations.

4. Rearrangements, inequalities, and convexity

The set of rearrangements of a function plays a role in this paper dictated by the physical considerations explained in § 1; it has nevertheless been an object of study in pure mathematics independently. Some parts of the resulting theory are presented here (without proofs) because they are needed in the proof of Theorem 1. Certain properties of the set of stratified rearrangements are deferred until the Appendix, since these are not standard and proofs must be given.

Because of the use of the convex set $\mathfrak{C}(q_0)$ in the proof of Theorem 1, we also give some theory of convex sets and weak convergence. Physically, one may view the elements of $\mathfrak{C}(q_0)$ as reflecting 'limiting' or 'averaged' properties of finely filamented (stratified) rearrangements of q_0; the good convergence properties of the Lebesgue integral ensure the existence of the required limits.

4.1. *General properties*

If f is integrable on a bounded measurable $\Omega \subset \mathbb{R}^N$, and g is a rearrangement of f on Ω, then g is integrable on Ω and

$$\int_\Omega f = \int_\Omega g.$$

If $f \in L^2(\Omega)$ and $g \in \mathscr{R}_\Omega(f)$ then g^2 is a rearrangement of f^2 and therefore $\|g\|_2 = \|f\|_2$. The convexity of $\|\cdot\|_2$ now ensures $\|g\|_2 \leqslant \|f\|_2$ for all $g \in \mathscr{C}_\Omega(f)$.

Consequently, if $q_0 \in L^2(\Omega)$ for bounded measurable $\Omega \subset \mathbb{R}^3$ then $\|q\|_2 = \|q_0\|_2$ for all $q \in \mathfrak{R}_\Omega(q_0)$, and $\|q\|_2 \leqslant \|q_0\|_2$ for all $q \in \mathfrak{C}_\Omega(q_0)$.

4.2. *Increasing rearrangements*

Any real integrable function f defined on a bounded measurable set $\Omega \subset \mathbb{R}^N$ has an *increasing rearrangement* f^* defined on the interval $(0, m)$ where m is the measure of Ω, which is an increasing function satisfying

$$|\{\xi \in (0,m) \colon f^*(\xi) \geqslant s\}| = |\{\boldsymbol{r} \in \Omega \colon f(\boldsymbol{r}) \geqslant s\}| \quad \text{for all } s > 0.$$

Then f^* is uniquely defined except for the values at its discontinuities.

If $f,\ g \in L^2(\Omega)$, then the inequality

$$\int_\Omega fg \leqslant \int_0^m f^* g^* \tag{5}$$

is classical; for a proof see for example Theorem 1 of Burton (1987*a*). From it may be deduced the inequality

$$\int_\Theta f \geqslant \int_0^\theta f^* \quad \text{for } \Theta \subset \Omega \text{ measurable, } \theta = |\Theta| \tag{6}$$

by setting $g(\boldsymbol{t}) = -1$ if $\boldsymbol{t} \in \Theta$, and $g(\boldsymbol{t}) = 0$ if $\boldsymbol{t} \in \Omega \setminus \Theta$.

Ryff (1965, Lemma 2), showed that any integrable function on an interval can be expressed as the composition of its increasing rearrangement with a measure-preserving transformation; see our Lemma 5 in the Appendix for further explanation.

4.3. *Steiner-symmetrization*

Any integrable function f defined on a symmetric interval $(-s, s) \subset \mathbb{R}$ has a *symmetric decreasing rearrangement* $f^\triangle$, that is a rearrangement as an even function on $(-s, s)$, decreasing on $(0, s)$. The inequality analogous to (5) holds for symmetric decreasing rearrangements, that is,

$$\int_{-s}^{s} fg \leqslant \int_{-s}^{s} f^{\triangle} g^{\triangle} \quad \text{for all } f, g \in L^2(-s, s). \tag{7}$$

If now $S := (-s, s) \times (-s, s) \times (-s, s)$ denotes a cube in $\mathbb{R}^3$ and $f \in L^1(S)$, then $f(\cdot, y, z)$ is an integrable function on $(-s, s)$ for almost every (y, z) in the square $Q := (-s, s) \times (-s, s)$; the *Steiner-symmetrization* f^s *of* f *in the* x*-direction* is defined to be such that $f^s(\cdot, y, z)$ is the symmetric decreasing rearrangement of $f(\cdot, y, z)$ for almost every $(y, z) \in Q$. From (7) we deduce

$$\int_S fg \leqslant \int_S f^s g^s \quad \text{for all } f, g \in L^2(S). \tag{8}$$

Steiner-symmetrization in the y-direction is similarly defined (we will not need it in the z-direction). A function that is invariant under Steiner-symmetrization in both the x- and y-directions will be called *doubly Steiner-symmetric*. The two operations of Steiner-symmetrization in the x- and y-directions do not commute. If however a function f is subjected to Steiner-symmetrization in both the x- and y-directions (in either order), the resulting rearrangement of f is doubly Steiner-symmetric.

4.4. *Riesz's inequality*

The notion of Steiner-symmetrization extends to certain non-negative functions on the whole of $\mathbb{R}^3$. Any function f that is admissible (in the sense of § 3.1) admits Steiner-symmetrizations; if f^s denotes its Steiner-symmetrization in the x-direction, then for almost every $(y, z) \in \mathbb{R}^2$ the function $f^s(\cdot, y, z)$ is the symmetric decreasing rearrangement of $f(\cdot, y, z)$. If f, g and h are admissible functions then a variant of *Riesz's inequality* asserts that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\boldsymbol{r}) g(\boldsymbol{r} - \boldsymbol{r}') h(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \leqslant \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^s(\boldsymbol{r}) g^s(\boldsymbol{r} - \boldsymbol{r}') h^s(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}', \tag{9}$$

where either side may be infinite. For a proof of (9), see Lieb & Loss (1997, Theorem 3.6). Clearly the above remarks apply also to Steiner-symmetrization in the y-direction.

4.5. *Consequences for energy functionals*

Suppose $q \in L^2(\mathbb{R}^3)$ is non-negative and has compact support. It follows from Riesz's inequality (9) that Steiner-symmetrization in either the x- or y-direction does not reduce $W(q)$ in (4). Steiner-symmetrization in the x-direction leaves $J(q)$ unchanged, whereas inequality (8) ensures that Steiner-symmetrization in the y-direction does not increase $J(q)$.

Consequently $E(q)$ is not reduced by Steiner-symmetrization in either the x- or y-directions. Thus q has a doubly Steiner-symmetric rearrangement $\bar{q}$ satisfying $E(\bar{q}) \geqslant E(q)$, i.e. the energy maximizer must be Steiner-symmetric.

4.6. *Convex sets and weak convergence*

As remarked in § 3.2, the solution for our variational problem on a bounded domain is obtained as a weak limit of a maximising sequence. However the sets $\mathscr{R}(q_0)$ and $\mathfrak{R}(q_0)$ are not in general weakly closed, so we may violate the constraints of the variational problem in passing to the weak limit. A first step towards overcoming this difficulty is to extend the constraint set. Therefore the sets $\mathscr{C}(q_0)$ and $\mathfrak{C}(q_0)$ are introduced because, as we explain below, closed convex sets behave well under weak convergence. With these considerations in mind, we review here some of the essentials of convex analysis, in the context of the Hilbert space $L^2(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^N$.

A set $C \subset L^2(\Omega)$ is called *convex* if C contains the straight line-segment joining each pair of its points. A convex set also contains all *convex combinations* of its points, i.e. the (finite) linear combinations whose coefficients are non-negative and sum to 1. The *closed convex hull* of a set $S \subset L^2(\Omega)$ is the intersection of all the (strongly) closed convex sets that contain S. If $M > 0$ and $\|x\|_2 \leqslant M$ for all $x \in S$, then the same inequality holds for all x lying in the closed convex hull of S. Hence the closed convex hull of a bounded set is also bounded.

Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ in $L^2(\Omega)$ *converges weakly* to $f \in L^2(\Omega)$ if

$$\int_\Omega f_n g \to \int_\Omega f g \text{ as } n \to \infty, \quad \text{for all } g \in L^2(\Omega).$$

The following one-dimensional example is illuminating: define $f_n(\xi) = \sin 2\pi n\xi$ for $\xi \in (0, 1)$. The f_n are all rearrangements of each other, and $f_n \to 0$ weakly as $n \to \infty$. In this case the weak limit is not a rearrangement. This construction therefore shows that the set of rearrangements of f_1 is not weakly closed in $L^2(0, 1)$.

Every closed convex set in $L^2(\Omega)$ contains the weak limits of all its weakly convergent sequences; this follows from Theorem 2.13 of Lieb & Loss (1997) for example. Thus a simple way to extend a set in $L^2(\Omega)$ to make it weakly closed, is to take its closed convex hull.

The existence of a weak limit for a maximising sequence of our variational problem uses a *weak compactness* argument. By contrast with the finite-dimensional situation, a bounded sequence in the infinite-dimensional space $L^2(\Omega)$ need not have a convergent sub-sequence. Indeed the functions f_n considered above illustrate this point, since $||f_n||_2 = 1/\sqrt{2}$ for every n, and $||f_m - f_n||_2 = 1$ for $m \neq n$. However, every bounded sequence in $L^2(\Omega)$ does have a sub-sequence converging *weakly* to some point of $L^2(\Omega)$; see for example Theorem 2.18 of Lieb & Loss (1997). It follows that if $C \subset L^2(\Omega)$ is closed, convex and bounded, then every sequence in C has a sub-sequence converging weakly to an element of C. This observation will allow us to prove the existence of an energy maximizer in the set $\mathfrak{C}_\Omega(q_0)$ introduced in § 3.1.

The functions in $\mathfrak{C}(q_0)$ may be regarded as limits of PV-fields in $\mathfrak{R}(q_0)$, taking the limit of highly filamented or vertically oscillating fields in a 'coarse-grained' sense (i.e. performing a local averaging of the PV-field), as in the statistical theory of Miller (1990) and Robert & Sommeria (1991). For our purposes, it is necessary that the energy maximizer in $\mathfrak{C}(q_0)$ in fact belongs to $\mathfrak{R}(q_0)$, i.e. that it is a true rearrangement of q_0 and not one of the 'coarse-grained' fields. This is shown with the help of Lemma 6 in the Appendix, and is a manifestation of the status of $\mathfrak{R}(q_0)$ as a set of 'vertices' of $\mathfrak{C}(q_0)$.

5. Preliminary estimates

We now perform some calculations of the energy and stream-function due to a stratified rearrangement of q_0 that will be used in the proof of Theorem 1.

Lemma 1. *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Then some stratified rearrangement q of q_0 with compact support satisfies $E(q) > 0$.*

Proof. Consider the rearrangement q of q_0 defined by $q(x, y, z) = q_0(\alpha x, \alpha^{-1}y, z)$ where $0 < \alpha \leqslant 1$. We make a linear change of variable to obtain

$$W(q) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q_0(\alpha x, \alpha^{-1}y, z)q_0(\alpha x', \alpha^{-1}y', z')\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'}{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}$$
$$= \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q_0(x, y, z)q_0(x', y', z')\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'}{(\alpha^{-2}(x-x')^2+\alpha^2(y-y')^2+(z-z')^2)^{1/2}}$$
$$= \frac{\alpha}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q_0(x, y, z)q_0(x', y', z')\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'}{((x-x')^2+\alpha^4(y-y')^2+\alpha^2(z-z')^2)^{1/2}} \geqslant \alpha W(q_0),$$

and

$$J(q) = \int_{\mathbb{R}^3}(c_0+c_1z)y^2q_0(\alpha x, \alpha^{-1}y, z)\mathrm{d}\boldsymbol{r}$$
$$= \int_{\mathbb{R}^3}(c_0+c_1z)(\alpha y)^2q_0(x, y, z)\mathrm{d}\boldsymbol{r} = \alpha^2J(q_0),$$

whence $E(q) \geqslant \alpha W(q_0) - \alpha^2J(q_0) > 0$ for sufficiently small α. □

Remark. The next lemma is adapted from Burton (1987*b* Lemma 4), and its proof makes use of the observation that if f is a non-negative decreasing function on $(0, \infty)$, then for $0 < \alpha < x$ we have

$$\int_{x-\alpha}^{x} f \leqslant \frac{\alpha}{x}\int_0^x f, \tag{10}$$

which is easily proved by a linear change of variables.

LEMMA 2. *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Then there is a positive constant C (depending on q_0 only) such that*

$$Kq(x, y, z) \leqslant C(x^2+y^2)^{-1/6} \quad \textit{whenever } x^2+y^2 \geqslant 2,$$

for every doubly Steiner-symmetric stratified rearrangement q of q_0.

Proof. Let ρ be the radius of the ball having the same volume as the set $\{\boldsymbol{r}' \in \mathbb{R}^3 \colon q_0(\boldsymbol{r}') > 0\}$. Let $\boldsymbol{r} = (x, y, z)$ and suppose $x^2 + y^2 = 2a^2$ where $a > 1$. Then $|x| \geqslant a$ or $|y| \geqslant a$; without loss of generality we assume $x \geqslant a$. Let $0 < b < a$. Fix a doubly Steiner-symmetric stratified rearrangement q of q_0. Then

$$Kq(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|}$$
$$= \frac{1}{4\pi}\left(\int_{|x'-x|>b} + \int_{|x'-x|<b}\right)\frac{q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|}$$
$$\leqslant \frac{1}{4\pi b}\int_{\mathbb{R}^3} q + \frac{1}{4\pi}\left(\int_{q(\boldsymbol{r}')>0}\frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|^2}\right)^{1/2}\left(\int_{|x'-x|<b} q^2(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)^{1/2}$$
$$\leqslant \frac{1}{4\pi b}\int_{\mathbb{R}^3} q + \frac{1}{4\pi}\left(\int_{|\boldsymbol{r}'-\boldsymbol{r}|<\rho}\frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|^2}\right)^{1/2}\left(\frac{b}{x}\int_{\mathbb{R}^3} q^2(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)^{1/2}$$
$$\leqslant \frac{1}{4\pi b}\|q\|_1 + \frac{(4\pi\rho)^{1/2}}{4\pi}\left(\frac{b}{a}\right)^{1/2}\|q\|_2\,,$$

where the Steiner-symmetry in x has been used in conjunction with (10) to derive the penultimate line. We now choose $b = a^{1/3}$ to obtain

$$Kq(\boldsymbol{r}) \leqslant C2^{-1/6}a^{-1/3} = C(x^2 + y^2)^{-1/6},$$

for some positive constant C depending only on q_0. □

LEMMA 3. *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Let a and γ be positive numbers. Then there is a positive number β such that for every stratified rearrangement q of q_0 satisfying $E(q) \geqslant \gamma$, there is a cube A of side a for which*

$$|\{\boldsymbol{r}' \in A \colon q(\boldsymbol{r}') > 0\}| \geqslant \beta.$$

Proof. Consider a positive number β, and suppose there exists a stratified rearrangement q of q_0 such that $E(q) \geqslant \gamma$, but

$$|\{\boldsymbol{r}' \in A \colon q(\boldsymbol{r}') > 0\}| < \beta$$

for every cube A of side a. We show that for a sufficiently small choice of β this leads to a contradiction. Let ρ denote the radius of the ball of volume β.

Fix $\boldsymbol{r} \in \mathbb{R}^3$ and let X denote a cube with centre $\boldsymbol{r}$ and side na, where n is a positive integer to be chosen later. Then X can be covered by cubes $A(1), \ldots, A(n^3)$ of side a. Hence

$$\begin{aligned}
Kq(\boldsymbol{r}) &= \frac{1}{4\pi}\left(\sum_{i=1}^{n^3}\int_{A(i)} + \int_{\mathbb{R}^3\setminus X}\right)\frac{q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}' - \boldsymbol{r}|} \\
&\leqslant \frac{1}{4\pi}\sum_{i=1}^{n^3}\left(\int_{A(i)} q^2(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)^{1/2}\left(\int_{\boldsymbol{r}'\in A(i),\, q(\boldsymbol{r}')>0}\frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}' - \boldsymbol{r}|^2}\right)^{1/2} + \frac{2}{4\pi na}\int_{\mathbb{R}^3\setminus X} q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \\
&\leqslant \frac{n^3}{4\pi}\left(\int_{\mathbb{R}^3} q^2(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)^{1/2}\left(\int_{|\boldsymbol{r}'-\boldsymbol{r}|<\rho}\frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}' - \boldsymbol{r}|^2}\right)^{1/2} + \frac{2}{4\pi na}\int_{\mathbb{R}^3} q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \\
&= n^3\rho^{1/2}(4\pi)^{-1/2}\|q\|_2 + 2(4\pi na)^{-1}\|q\|_1.
\end{aligned}$$

Consequently

$$W(q) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q(\boldsymbol{r})Kq(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \leqslant 2^{-2}n^3\pi^{-1/2}\rho^{1/2}\|q\|_2\|q\|_1 + (4\pi na)^{-1}\|q\|_1^2.$$

We now choose n large enough to ensure $(4\pi na)^{-1}\|q_0\|_1^2 < \gamma/2$ and then choose β (and therefore ρ) small enough to ensure $2^{-2}n^3\pi^{-1/2}\rho^{1/2}\|q_0\|_2\|q_0\|_1 < \gamma/2$, choices that depend on a, γ and q_0 but not on the particular rearrangement q. We find that $E(q) \leqslant W(q) < \gamma$, and this contradiction shows that β has the desired properties. □

LEMMA 4. *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and vanish outside a cube of side α and centre $\boldsymbol{o}$. Let a, β be positive numbers, $a < \alpha$. Then there is a positive number δ such that, if q is any doubly Steiner-symmetric stratified rearrangement of q_0 satisfying*

$$|\{\boldsymbol{r}' \in A \colon q(\boldsymbol{r}') > 0\}| \geqslant \beta \tag{11}$$

for some cube A of side a, then

$$|\{(x, y) \in \mathbb{R}^2 \colon Kq(x, y, z) - (c_0 + c_1 z)y^2 > \delta\}| > \alpha^2 \quad \textit{for all } z \in [-\alpha, \alpha].$$

Proof. Consider a cube A of side a and a doubly Steiner-symmetric stratified rearrangement q of q_0 satisfying (11). Since $q_0(x,y,z)$ vanishes when $|z| > \alpha$, there is no loss of generality in assuming A lies in the region defined by $-\alpha < z < \alpha$. Moreover, the symmetry of q ensures that symmetrizing A in the x- and y-directions does not reduce the volume in (11); we may therefore assume A is centred on the z-axis.

Suppose $\boldsymbol{r} = (x,y,z)$ with $|z| < \alpha$. Then, using (6),

$$
\begin{aligned}
Kq(\boldsymbol{r}) &\geqslant \frac{1}{4\pi}\int_A \frac{q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|} \\
&\geqslant \frac{1}{4\pi}((|x|+a)^2+(|y|+a)^2+4\alpha^2)^{-1/2}\int_A q(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \\
&\geqslant \frac{1}{8\pi}(x^2+y^2)^{-1/2}\int_0^\beta q_0^* \;=:\; \kappa(x^2+y^2)^{-1/2}
\end{aligned}
$$

say, provided that $(x^2+y^2)^{1/2} \geqslant \xi := (2a^2+4\alpha^2)^{1/2}$, where * denotes increasing rearrangement onto the real interval $(0,v)$ with $v = |\{\boldsymbol{r}' \in \mathbb{R}^3 \colon q_0(\boldsymbol{r}') > 0\}|$. Therefore

$$
Kq(\boldsymbol{r}) - (c_0+c_1\alpha)y^2 \geqslant \kappa(x^2+y^2)^{-1/2} - (c_0+c_1\alpha)y^2
$$

whenever $(x^2+y^2)^{-1/2} > \xi$. Now the planar region defined by the inequality

$$
\kappa(x^2+y^2)^{-1/2} - (c_0+c_1\alpha)y^2 > 0
$$

has infinite area, because it contains the region of infinite area defined by the inequalities

$$
0 < y < x, \quad y < \kappa^{1/2}2^{-1/2}(c_0+c_1\alpha)^{-1/2}x^{-1/2}.
$$

We can therefore choose $\delta > 0$ such that the region defined by

$$
\kappa(x^2+y^2)^{-1/2} - (c_0+c_1\alpha)y^2 > \delta, \quad x^2+y^2 > \xi^2
$$

has area at least α^2. Then

$$
|\{(x,y)\colon Kq(x,y,z) - (c_0+c_1 z)y^2 > \delta\}| > \alpha^2
$$

for $|z| < \alpha$, where $\delta > 0$ depends on q_0 but not on q. $\square$

6. Proof of Theorem 1

In this section we give a rigorous proof of our main existence theorem, using the results proved in Lemmas 1–7. (Lemmas 5–7 are in the Appendix.) The main ideas of the proof are described in § 3.2.

Consider a rectangular domain $\Omega = Q \times I$ where Q is a square centred at the origin in the (x,y)-plane and $I = [z_0, z_1]$. Choose $\alpha \geqslant 2z_1$ so that if Q has side at least α then Ω contains the support of q_0, and define

$$
e = \sup\{E(q)\colon q \in \mathfrak{C}_\Omega(q_0)\}.
$$

Let $\{q_n\}_{n=1}^\infty$ be a maximizing sequence, that is, a sequence in $\mathfrak{C}_\Omega(q_0)$ for which $E(q_n) \to e$. Now $\mathfrak{C}_\Omega(q_0)$ is a bounded set in the Hilbert space $L^2(\Omega)$, hence $\{q_n\}_{n=1}^\infty$ has a subsequence $\{q_{n_j}\}_{j=1}^\infty$ that converges weakly to some limit $\bar{q} \in L^2(\Omega)$. Since $\mathfrak{C}_\Omega(q_0)$

is closed and convex, $\bar{q} \in \mathfrak{C}_\Omega(q_0)$. The compactness of K as a linear operator on $L^2(\Omega)$ (which follows from the square-integrability of $|\boldsymbol{r} - \boldsymbol{r}'|^{-1}$ over $\Omega \times \Omega$) ensures that $Kq_{n_j} \to K\bar{q}$ strongly in $L^2(\Omega)$, hence $E(q_{n_j}) \to E(\bar{q})$ as $j \to \infty$, and therefore $E(\bar{q}) = e$. This proves the existence of a maximizer $\bar{q}$ of E relative to the extended class of functions $\mathfrak{C}_\Omega(q_0)$.

To derive the first-variation condition satisfied by $\bar{q}$, we use the strict convexity of E. Consider any $q \in \mathfrak{C}_\Omega(q_0)$, $q \neq \bar{q}$. Then we have

$$E(\bar{q}) \geqslant E(q) = E(\bar{q}) + \int_\Omega (q(\boldsymbol{r}) - \bar{q}(\boldsymbol{r}))(K\bar{q}(\boldsymbol{r}) - (c_0 + c_1 z)y^2)\mathrm{d}\boldsymbol{r} + W(q - \bar{q})$$

$$> E(\bar{q}) + \int_\Omega (q - \bar{q})\Psi,$$

where $\Psi(\boldsymbol{r}) = K\bar{q}(\boldsymbol{r}) - (c_0 + c_1 z)y^2$, hence

$$\int_\Omega q\Psi < \int_\Omega \bar{q}\Psi.$$

This shows that $\bar{q}$ is the unique maximizer relative to $\mathfrak{C}_\Omega(q_0)$ of the bounded linear functional

$$q \mapsto \int_\Omega q\Psi.$$

Since Lemma 6 assures us that the supremum of any bounded linear functional relative to $\mathfrak{C}_\Omega(q_0)$ is attained by at least one element of $\mathfrak{R}_\Omega(q_0)$ we can deduce that $\bar{q} \in \mathfrak{R}_\Omega(q_0)$. Lemma 7 provides a function $\varphi\colon \mathbb{R}^2 \to [0, \infty)$ such that $\bar{q}(x, y, z) = \varphi(\Psi(x, y, z), z)$ almost everywhere in Ω, and $\varphi(\cdot, z)$ is increasing for almost every z. Thus, our maximizer relative to the extended set of functions $\mathfrak{C}_\Omega(q_0)$ turns out to be a stratified rearrangement, and is an increasing function of Ψ for almost every fixed z.

The above argument was conducted on a bounded domain $\Omega = Q \times I$, and in principle $\bar{q}$ could depend on the choice of Q. We now proceed to show that if Q is chosen large enough, it ceases to have any influence whatever on the problem. This is achieved using the estimates developed in § 5, which are Q-independent. We begin by recalling our observation in § 4.5 that Steiner-symmetrization of q in either the x- or y-directions does not reduce $E(q)$. We therefore assume that $\bar{q}$ is doubly Steiner-symmetric.

By Lemma 1 we can choose $l \geqslant \alpha \geqslant 2z_1$ and $\gamma > 0$ such that if Q has side at least l then $e \geqslant \gamma$. Next an application of Lemmas 3 and 4 shows that $\delta > 0$ may be chosen, independent of Q (having side at least l), such that $\Psi(\cdot, z) > \delta$ occurs on a set of area at least α^2, for every $z \in I$. Now the estimate of Lemma 2 shows that if $(x, y, z) \in \mathbb{R}^3$ and $\Psi(x, y, z) > \delta$ then $x^2 + y^2 \leqslant \max\{2, (C/\delta)^6\}$, where C is independent of Q. Let Q_0 denote the square whose side is $\max\{\sqrt{2}, l, (C/\delta)^3\}$, let $\Omega_0 = Q_0 \times I$, and henceforth assume Q is bigger than Q_0. Since, for almost every $z \in I$, the subset of Q where $\bar{q}(\cdot, z) > 0$ has area at most α^2, and $\bar{q}(\cdot, z)$ is equal almost everywhere in Q to an increasing function of $\Psi(\cdot, z)$, it follows that $\bar{q}(\cdot)$ is positive only at points of Ω where $\Psi(\cdot) > \delta$, except for a set of zero volume.

Thus, under our assumption that Q is larger than Q_0, it follows that $\bar{q}$ vanishes outside Ω_0. Hence if we take $\bar{q}$ to be the maximizer for Ω_0, then $\bar{q}$ maximises E over all stratified rearrangements of q_0, no matter how large their supports may be. The corresponding φ can now be extended so that $\varphi(u, z) = 0$ if $u \leqslant \delta$ or $z \notin I$; then each $\varphi(\cdot, z)$ is increasing, and $\bar{q} = \varphi(\Psi, z)$ almost everywhere in $\mathbb{R}^3$. $\square$

7. Discussion

In Theorem 1 we have proved the existence of a stationary vortex solution of equation (3) in the set of stratified rearrangements of any given PV-anomaly field q_0 (i.e. in any family of isovortical flows) that satisfies the following conditions: q_0 must have compact support, it must have the same sign everywhere, and this sign must be the same as the sign of the background shear $2(c_0 + c_1 z)$ over the interval in z to which the support of q_0 is confined. If $q_0 \geqslant 0$ the PV-field of the maximizer is symmetric decreasing in x and y for every fixed z (symmetric increasing if $q_0 \leqslant 0$).

The fact that a flow maximises the energy implies that it is linearly stable (Nycander 1995). It should also mean that the flow is nonlinearly stable in a practical sense, as argued by Benjamin (1976). This is analogous to Lyapunov stability for a system with a finite number of degrees of freedom. However, we cannot formalize this to a statement of stability in some norm.

In one case the shape of the stationary vortex can be found analytically. If the PV-anomaly is constant inside an ellipsoidal surface, and vanishes outside this surface, and if the stream function of the background flow is a quadratic function, then the discontinuity surface will always remain ellipsoidal, and the general time-dependent solution can be found (Meacham *et al.* 1994). Steady solutions of this kind can be found both in adverse shear and cooperative shear, and the present result suggests that those in cooperative shear are stable.

One possible generalization of Theorem 1 is to add a term $-(d_0 + d_1 z)x^2$ to the stream function of the background flow, which is then a general strain flow. (This can be a simple model of the flow induced by other vortices in the vicinity, employing a reference frame in which the strain axes are fixed.) In accordance with the heuristic argument of § 2, we expect an energy maximizer to exist if this stream function is sign-definite, i.e. if the origin is an elliptic stagnation point of the background flow. If the origin is a hyperbolic stagnation point, on the other hand, the external potential of our heuristic argument has no minimum, and it is clear that no maximizer exists.

This picture agrees very well with the the analysis by Dritschel & Torre Juárez (1996) of the linear stability of a stationary vortex column (i.e. an elliptic cylinder with uniform PV-anomaly) in an external strain flow. They find that if the vorticity of the external flow is larger than or equal to the strain rate, and has the same sign as the PV-anomaly of the vortex, such a vortex is stable. (This is exactly the parameter region in which an energy maximizer should exist.) In all other cases the vortex column is linearly unstable, provided that the height of the atmosphere is large enough to accommodate the unstable mode.

Dritschel & Torre Juárez (1996) also performed nonlinear simulations of unstable vortex columns. They found that a freely rotating elliptic vortex column (i.e. without external flow) either axisymmetrized, while shedding PV-filaments, or performed reversible oscillations. In the presence of an external strain field, on the other hand, tall vortices were disrupted, resulting in compact vortices with an aspect ratio (height to mean radius) of about 3. The fact that such vortices are disrupted is not surprising, since they correspond to saddle points of the energy, rather than maximum points.

Another generalization is to add a constant vertical shear to the background flow of equation (3). Provided that $c_1 \neq 0$, this case can be recovered from equation (3) by a shift of the coordinate system in the y-direction, plus a Galileian transformation in the x-direction. In effect, this means that the energy maximizer is a vortex located at the y-value where the vertical shear vanishes, and travelling with the background flow velocity at this position.

In our model we assumed that there are no vertical boundaries. Often, however, equation (2) is solved with the boundary conditions $\partial\Psi/\partial z = 0$ at $z = 0$ and $z = H$. We believe that the corresponding existence theorem is true for this bounded case as well, but we have not been able to prove this. The problem is that the energy functional is not convex in the bounded case. Thus, the theorem proved here is relevant only for vortices that are small (both vertically and horizontally) compared to the total height of the atmosphere. Note, however, that the two-dimensional case may be considered as the opposite limit, with the vortex occupying the whole height of the atmosphere, and the horizontal size much larger than this height, and in this case the corresponding existence theorem holds (Nycander 1995).

An important difference between the two cases is that a stationary vortex has a separatrix in the unbounded case considered here, but not in the bounded case. The reason is that the Green's function diverges logarithmically at infinity in the bounded case, while it behaves as $1/|\boldsymbol{r}|$ in the case considered here.

As shown in Lemma 1, the energy maximizer must have positive energy. This makes it possible to estimate the amplitude necessary for a stationary vortex to have approximately spherical shape, as opposed to a strongly elongated shape. If we assume, for simplicity, that $q = q_0 =$ const. inside and $q = 0$ outside the sphere $|\boldsymbol{r}| < a$, and that the background flow is independent of z (i.e. that $c_1 = 0$), it is straightforward to calculate that the energy is $E = (2/15)\pi q_0 a^5(2q_0 - c_0)$. The first term represents the perturbation energy W and the second term the external contribution J in equation (4). Hence, if $q_0 < c_0/2$ the stationary vortex must be significantly elongated, and is therefore probably less robust than if $q_0 > c_0/2$. (The perturbation energy W can be thought of as a 'binding energy' of the vortex.) This crude estimate is perhaps supported by the observation in the turbulence simulations by McWilliams (1989) that coherent vortices emerge in regions where the vorticity is larger than the local strain rate.

The realization that stable vortices are maximum energy states can be an important tool for the intuitive thinking about vortex dynamics, apart from its role in the present existence proof. For instance, numerical simulations of three-dimensional quasi-geostrophic flow have revealed a tendency for vortices to align vertically (if they have the same sign) and to axisymmetrize horizontally (McWilliams 1989; Viera 1995; Sutyrin *et al.* 1996). Both these processes can be interpreted as a tendency to approach the maximum energy state, which is a vertically aligned axisymmetric vortex (in the absence of background flow).

It is typical for many nonlinear infinite-dimensional systems that conditional extrema of conserved quantities act as attractors in this way. In dissipative systems this is often interpreted as a 'selective decay' of the invariants. In the ideal model used here, the conservation of PV and energy of course prevents an unsteady flow from evolving into a maximum-energy state. However, the excitation of small scales (i.e. filamentation of the PV-field) can effectively act like dissipation, and in a course-grained sense move the flow to a different isovortical family where it is close to a maximum-energy state. (We note that this process still keeps the PV-field in the same closed convex hull, the set $\mathfrak{C}(q_0)$ discussed in § 3.1 and § 4.6.) This is the basic idea behind the statistical mechanical theory for ideal two-dimensional flow of Miller (1990) and Robert & Sommeria (1991). It seems likely that this theory can be generalized to the model studied in the present article.

The vertical alignment and horizontal axisymmetrization are irreversible, nonlinear processes. However, it has also been observed in simulations that columns of uniform PV can perform a reversible and almost periodic motion (Viera 1995; Sutyrin *et* *al.* 1996; Dritschel & Ambaum 1997). This may be interpreted as a large-amplitude extension of a linear wave on the axisymmetric stationary state. The dispersion relation for the linear waves is $\omega = mQ(1/2 - I_m(ka)K_m(ka))$, where Q is the PV and a the radius of the column, I_m and K_m modified Bessel functions, and m and k the azimuthal and vertical wavenumbers, respectively. The nonlinear, irreversible behaviour sets in only if the wave amplitude (i.e. the deviation from the axisymmetric state) is large enough, as studied in detail by Sutyrin *et al.* (1996).

We caution, however, that a column of uniform PV can probably tolerate oscillations of larger amplitude before the nonlinear behaviour sets in than smoother vortices, as is the case in two-dimensional flow (Dritschel 1998). If, for example, the PV of the vortex column is a strictly decreasing (or strictly increasing) function of r, no normal modes exist, as can be shown similarly to Appendix B of Åkerstedt, Nycander & Pavlenko (1996). This means that any infinitesimal perturbation will be sheared away, and that the vortex approaches axisymmetry as $t \to \infty$ even in the linear approximation. For two-dimensional flow, a proof of this linear asymptotic behaviour was recently given by Bassom & Gilbert (1998).

In numerical simulations of three-dimensional quasi-geostrophic turbulence that use the boundary conditions $\partial\Psi/\partial z = 0$ at $z = 0$ and $z = H$, a very clear preference is seen for coherent vortices to form at the top and at the bottom of the domain (McWilliams 1989; Dritschel & Ambaum 1997). This can be understood in terms of the maximum-energy argument employed in the present work. Poisson's equation can in this case be solved by introducing mirror vortices outside the boundaries of the domain, with the same sign as the real vortices. If a vortex touches the boundary it also touches a mirror vortex, in effect forming a 'virtual vortex' twice the size of the real vortex. The energy is therefore much larger than if the real vortex were situated in the middle of the domain. This makes vortices at the boundaries more robust.

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Appendix. The space of stratified rearrangements

The proof of Theorem 1 required extensions of some standard results on spaces of rearrangements to the stratified case. Here we give the details. The issue that arises is whether the operations we perform at each z-level fit together in a measurable way. Lemmas 6 and 7 are the stratified counterparts of Theorems 4 and 5 of Burton (1987*a*); we have taken the opportunity to simplify the proofs.

We begin with a result that was proved by Ryff (1965, Lemma 2), for functions on an interval. We omit the proof, since Ryff's argument carries over to our case with only a slight modification, concerning level sets having positive area. We indicate the necessary modification by giving a formula in the statement of Lemma 5.

If $\Theta \subset \mathbb{R}^N$ is a bounded measurable set and $\theta = |\Theta|$, a map $\sigma\colon \Theta \to (0, \theta)$ is called *measure-preserving* if $|\{\boldsymbol{t} \in \Theta : \sigma(\boldsymbol{t}) \leqslant \xi\}| = \xi$ for every $0 < \xi < \theta$. If σ is a measure-preserving map then $|\sigma^{-1}(B)| = |B|$ for every measurable set $B \subset (0, \theta)$. A measure-preserving map need not be invertible.

Lemma 5. *Let $U \subset \mathbb{R}^2$ be a bounded measurable set, with $|U| = m$ say, and let f be* *a real integrable function on U. For $(x,y)\in U$ define*

$$\sigma(x,y)=|\{(x',y')\in U\colon f(x',y')<f(x,y)\}|$$
$$+|\{(x',y')\in U\colon f(x',y')=f(x,y) \text{ and } x'<x\}|$$

Then $\sigma\colon U\to[0,m]$ is a measure-preserving map and $f=f^*\circ\sigma$ almost everywhere in U.

In the next lemma the function ψ defines a general bounded linear functional on q, and it is shown that the supremum of such a functional relative to $\mathfrak{C}(q_0)$ is attained by an element in $\mathfrak{R}_\Omega(q_0)$. We have already suggested thinking of the elements of $\mathfrak{R}(q_0)$ as 'vertices' of $\mathfrak{C}(q_0)$; Lemma 6 tells us that every 'face' of $\mathfrak{C}(q_0)$ contains such a 'vertex'.

LEMMA 6. *Let $\Omega=Q\times I$ be a rectangular domain where $Q=(-\alpha,\alpha)\times(-\alpha,\alpha)\subset\mathbb{R}^2$ and $I=(z_0,z_1)\subset\mathbb{R}$. Let $q_0\in L^2(\Omega)$ and $\psi\in L^2(\Omega)$, and let $q_0^*(\cdot,z)$ and $\psi^*(\cdot,z)$ be the increasing rearrangements of $q_0(\cdot,z)$ and $\psi(\cdot,z)$ respectively on $[0,\alpha^2]$, which exist for almost every $z\in I$. Then there is a measurable function $\sigma\colon\Omega\to[0,\alpha^2]$ such that for almost every $z\in I$, the map $\sigma(\cdot,z)\colon Q\to[0,\alpha^2]$ is measure-preserving, and $\psi(\cdot,z)=\psi^*(\sigma(\cdot,z),z)$ almost everywhere in Q.*

Further $\tilde q(x,y,z):=q_0^*(\sigma(x,y,z),z)$ for $(x,y,z)\in\Omega$ defines $\tilde q\in\mathfrak{R}_\Omega(q_0)$ that realizes the supremum of $\int_\Omega q\psi$ relative to $\mathfrak{C}_\Omega(q_0)$.

Proof. For almost every $z\in I$, we have $q_0(\cdot,z)$, $\psi(\cdot,z)\in L^2(Q)$, and for any rearrangement χ of $q_0(\cdot,z)$ we have

$$\int_Q \chi(x,y)\psi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\leqslant\int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)\,\mathrm{d}t; \tag{12}$$

note that $q_0^*,\psi^*\in L^2((0,\alpha^2)\times I)$. The left-hand side of (12) defines a bounded linear functional of χ; the inequality (12) therefore holds when χ belongs to the closed convex hull of the rearrangements of $q_0(\cdot,z)$. Now taking $q\in\mathfrak{C}_\Omega(q_0)$ we can set $\chi=q(\cdot,z)$ in (12) and integrate with respect to z to obtain

$$\int_\Omega q\psi\leqslant\int_{z_0}^{z_1}\int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)\,\mathrm{d}t\,\mathrm{d}z\quad\text{for all } q\in\mathfrak{C}_\Omega(q_0). \tag{13}$$

We now construct $q\in\mathfrak{R}_\Omega(q_0)$ that realizes equality in (13). Define

$$\sigma(x,y,z)=|\{(x',y')\colon\psi(x',y',z)<\psi(x,y,z)\}|$$
$$+|\{(x',y')\colon x'<x \;\text{ and }\; \psi(x',y',z)=\psi(x,y,z)\}|.$$

Then $\sigma\colon\Omega\to[0,\alpha^2]$ is a measurable function. Moreover Lemma 5 assures us that for almost every fixed z, the map $\sigma(\cdot,z)$ is measure-preserving and satisfies $\psi(\cdot,z)=\psi^*(\sigma(\cdot,z),z)$. If we choose $\tilde q(\cdot,z)=q_0^*(\sigma(\cdot,z),z)$ then $\tilde q\in\mathfrak{R}_\Omega(q_0)$, and for almost every z,

$$\int_Q\tilde q(x,y,z)\psi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\alpha^2}q_0^*(t,z)\psi^*(t,z)\,\mathrm{d}t.$$

Now integrating with respect to z yields equality in (13) as desired. □

LEMMA 7. *Let $\Omega=Q\times I$ be a rectangular domain where $Q=(-\alpha,\alpha)\times(-\alpha,\alpha)\subset\mathbb{R}^2$ and $I=(z_0,z_1)\subset\mathbb{R}$. Let $q_0\in L^2(\Omega)$ and $\psi\in L^2(\Omega)$. Suppose $\int_\Omega q\psi$ attains its maximum relative to $\mathfrak{R}_\Omega(q_0)$ at a unique element $\bar q$. Then there is a real function φ defined on $\mathbb{R}\times I$ such that $\bar q(x,y,z)=\varphi(\psi(x,y,z),z)$ for almost every $(x,y,z)\in\Omega$, and such that $\varphi(\cdot,z)$ is increasing for almost every $z\in I$.*

Proof. Let ψ^*, q_0 and σ be as in Lemma 6. Then $\psi(x,y,z) = \psi^*(\sigma(x,y,z),z)$ and, by uniqueness and Lemma 6, $\bar{q}(x,y,z) = q_0^*(\sigma(x,y,z),z)$, for almost every $(x,y,z) \in \Omega$.

Now for almost every $z \in I$, the functions $q_0^*(\cdot,z)$ and $\psi^*(\cdot,z)$ are increasing on $[0,\alpha^2]$. In order to show that $q_0^*(\cdot,z)$ is almost everywhere an increasing function of $\psi^*(\cdot,z)$, it will be enough to show that on any open interval where $\psi^*(\cdot,z)$ is constant, $q_0^*(\cdot,z)$ is constant also, for almost every $z \in I$.

Consider rational numbers $r < s$ and let $Z(r,s)$ denote the set of $z \in I$ such that $\psi^*(\cdot,z)$ is constant on the open interval (r,s) but $q_0^*(\cdot,z)$ is non-constant on (r,s). Then $Z(r,s)$ is measurable; we show $Z(r,s)$ has measure zero. Consider the possibility that $Z(r,s)$ has positive measure. Define

$$\hat{q}(t,z) = \begin{cases} q_0^*(r+s-t,z) & \text{if } t \in (r,s) \text{ and } z \in Z(r,s), \\ q_0^*(r,s) & \text{if } t \notin (r,s) \text{ or } z \notin Z(r,s). \end{cases}$$

Then, for almost every $z \in I$, $\hat{q}(\cdot,z)$ is a rearrangement of $q_0^*(\cdot,z)$. Hence

$$q_1(x,y,z) = q_0^*(\sigma(x,y,z,),z) \text{ for all } (x,y,z) \in \Omega$$

defines $q_1 \in \Re_\Omega(q_0)$, and moreover the constancy of $\psi(\cdot,z)$ on (r,s) for $z \in Z(r,s)$ ensures that

$$\int_\Omega q_1\psi = \int_{z_0}^{z_1}\int_0^{\alpha^2} \hat{q}(t,z)\psi(t,z)\mathrm{d}t\,\mathrm{d}z = \int_{z_0}^{z_1}\int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)\mathrm{d}t\,\mathrm{d}z = \int_\Omega \bar{q}\psi.$$

But $q_0^*(\cdot,z)$ is increasing and non-constant on (r,s) for all $z \in Z(r,s)$ hence $\hat{q}$ differs from q_0^* on a set of positive measure, hence q_1 differs from $\bar{q}$ on a set of positive measure. This contradicts the uniqueness of the maximizer $\bar{q}$. Hence $Z(r,s)$ has zero measure as desired.

Now let

$$Z = \bigcup_{r,\,s\in\mathbb{Q},\,r<s} Z(r,s)$$

which has zero measure, being a countable union of sets of zero measure (here $\mathbb{Q}$ denotes the set of all rational numbers). Consider $z \in I \setminus Z$. If $\psi^*(\cdot,z)$ is constant on an interval (p,q), then $q_0^*(\cdot,z)$ is constant on (r,s) for all rationals r and s with $p < r < s < q$, hence $q_0^*(\cdot,z)$ is constant on (p,q). Therefore $q_0^*(\cdot,z) = \varphi(\psi^*(\cdot,z),z)$ almost everywhere on $[0,\alpha^2]$ for some increasing function $\varphi(\cdot,z)$; then by composing with σ we obtain $\bar{q}(\cdot,z) = \varphi(\psi(\cdot,z),z)$ almost everywhere on Q.

Thus $\bar{q}(x,y,z) = \varphi(\psi(x,y,z),z)$ for almost all $(x,y,z) \in \Omega$ as desired. □

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